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Curves of marginal stability in two-dimensional $CP(N-1)$ models with Z_N-symmetric twisted masses

S Ölmez[1] **and M Shifman**[2]

[1] Department of Physics and Astronomy, University of Minnesota, Minneapolis, MN 55455, USA
[2] William I. Fine Theoretical Physics Institute, University of Minnesota, Minneapolis, MN 55455, USA

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Abstract

We consider curves of marginal models in two dimensions with $\mathcal{N} = (2, 2)$ supersymmetry. In these theories, one can introduce twisted mass terms. The BPS spectrum has different numbers of states in the weak and strong coupling regimes. This spectral restructuring can be explained by the fact that two regimes are separated by CMS on which some BPS states decay. We focus on a special case of Z_N-symmetric twisted masses. In this case, the general solution due to Dorey greatly simplifies, and CMS can be found explicitly. For small-N values numerical results are presented. In the large-N limit, we find CMS analytically up to $\ln N/N$ corrections.

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1. Introduction

Two-dimensional $CP(N-1)$ sigma models with $\mathcal{N} = (2, 2)$ supersymmetry present a rich theoretical laboratory. In addition to the scale constant Λ, one can introduce other dimensional parameters, the so-called twisted masses, which can be interpreted as expectation values of a background twisted chiral multiplet [1, 2]. An exact description of the spectrum of the Bogomol'nyi–Prasad–Sommerfield (BPS) states as a function of the twisted masses is presented in [3]. The spectrum of the theory with nonzero twisted masses includes 'dyons' in shortened multiplets. The dyon carries both the topological and the Noether charges. The dyon mass is given by the absolute value of the sum of the topological mass, $m_{\mathcal{T}}$, and the Noether mass, $m_{\mathcal{N}}$, which are complex parameters,

$$M = |m_{\mathcal{T}} + m_{\mathcal{N}}|. \tag{1}$$

The triangular inequality for complex numbers gives $M \leqslant |m_{\mathcal{T}}| + |m_{\mathcal{N}}|$. If the equality is satisfied,

$$M = |m_{\mathcal{T}}| + |m_{\mathcal{N}}|, \tag{2}$$

this is a boundary situation of a spectral restructuring. A submanifold in the parameter space, where the equality is satisfied is called CMS. If one crosses this manifold, discontinuities appear in the spectrum. CMS, and the corresponding discontinuities of the BPS spectrum, appear in theories with centrally extended supersymmetry algebras [4]. A detailed analysis of metamorphosis of the BPS spectrum in the neighborhood of CMS is given in [5]. Dimension of the submanifold, determined by the condition of the marginal stability, need not be one. It can be larger depending on the number of degrees of freedom residing in the twisted masses. First we must note that the only condition on the twisted masses is

$$\sum_{l=0}^{N-1} m_l = 0. \tag{3}$$

This means that we have $2(N-1)$ real independent parameters, for arbitrary N. However, we will limit ourselves to a very special and very interesting case of Z_N-symmetric masses,

$$m_l = m\,\mathrm{e}^{\frac{2\pi i l}{N}}, \qquad l = 0, 1, \ldots, N-1. \tag{4}$$

(Why it is of special physical interest is explained in [6].) If one introduces the masses Z_N symmetrically, one has only two independent real parameters, which come from the complex parameter m, for any value of N. It is important to note that equation (3) is automatically satisfied. The condition of CMS, equation (2), reduces the number of independent parameters on CMS from two to one; thus, in this case CMS are indeed curves in the complex m-plane. This is another reason to consider the theory with Z_N-symmetric twisted masses. A general consideration of CMS in the $CP(N-1)$ model is presented in [3]. For $N = 2$, the explicit form of CMS is found in [7]. In this paper, we will consider $CP(N-1)$ sigma model with Z_N-symmetric twisted masses and arbitrary N. We will find explicit equations for CMS for any N. We give numerical solutions for small N, and show that for large N CMS are circles with

$$|m| = e\Lambda. \tag{5}$$

The organization of the paper is as follows. In section 2, we introduce the Z_N-symmetric twisted masses and derive the equation determining CMS. We solve this equation numerically for $N = 4$. In section 3, we determine CMS explicitly for large values of N. The notation and a brief introduction for our framework are given in the appendix. In the first part of the appendix, we briefly introduce $\mathcal{N} = 2$ supersymmetric field theories in two dimensions (for details see [8]). In the second part, we discuss the central extension of the theory and introduce the mirror representation. Finally, we consider $CP(N-1)$ models with twisted masses and derive the conditions on the twisted masses to produce CMS.

2. Z_N-symmetric masses

In the Z_N-symmetric case, the form of the twisted masses is given in equation (4), see figures 1 and 2, which show Z_N-symmetric masses for $N = 3$ and $N = 4$, respectively. The twisted masses are measured in the units of the scale constant Λ which is set to 1 (see the appendix for details). Figures 1 and 2 are plotted at $m = \mathrm{e}$. In the general case, $m = \mu\,\mathrm{e}^{\mathrm{i}\theta}$ the scale of the corresponding plots changes from e to μ and they are rotated counterclockwise by the angle θ.

The bosonic part of the $CP(N-1)$ model deformed by twisted masses can be written as

$$S = \frac{2}{g^2}\int \mathrm{d}^2x \left\{ |(\partial_\alpha - \mathrm{i}A_\alpha)n^l|^2 + \sum_l |(\eta - m_l)n^l|^2 \right\}, \tag{6}$$

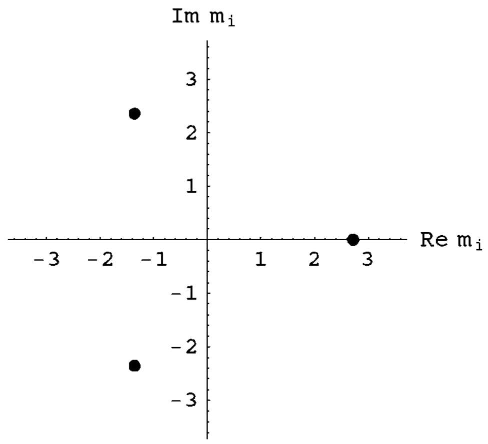

Figure 1. Z_3-symmetric masses in the complex m plane.

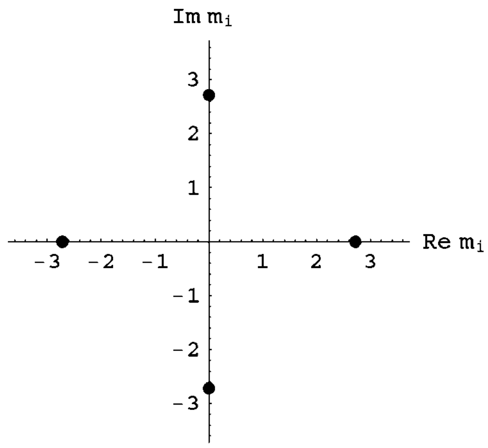

Figure 2. Z_4-symmetric masses in the complex m plane.

with the condition $\sum_{l=0}^{N-1} \bar{n}^l n_l = 1$ [6]. A_α and η are auxiliary fields, which have no kinetic terms. n^l $(l = 0, 1, \ldots, N-1)$ is a complex field. The second term in the action represents the twisted mass deformation. Equation (6) gives the action of the $CP(N-1)$ sigma model in the linear gauged representation. The fermionic part of the action can be constructed by requiring that the action is $\mathcal{N} = 2$ supersymmetric [9]. The theory can be solved in $1/N$ expansion for large values of N [10] (for nonvanishing twisted masses see [6]). The theory has N vacua, and for each vacuum only one of the n^l fields is nonvanishing. For example, say, for the kth vacuum we can set $\eta = m_k$ so that the $l = k$ term in the sum vanishes. For the other terms to vanish we require $n^l = 0$ for $l \neq k$. If the twisted masses are of the form given by equation (4), then the action has an apparent Z_N symmetry. This symmetry is spontaneously broken, as the vacuum is not Z_N symmetric.

Although the representation given in equation (6) is very transparent, it is not convenient for our purposes. It is more convenient for us to work in the *mirror representation* which is described in the appendix.

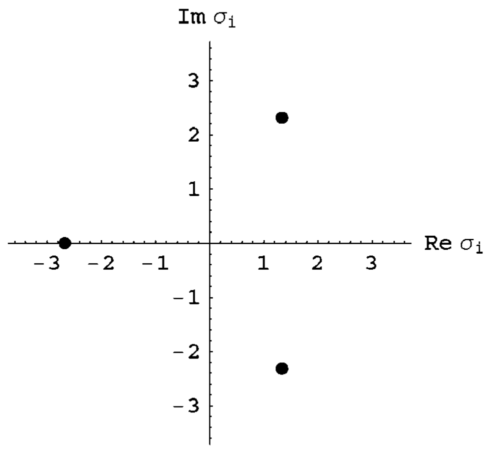

Figure 3. Z_3-symmetric vacua in the complex σ-plane.

Hori and Vafa, who originally suggested the mirror representation, derived it in the form of the Toda chain. Since then a few other equivalent representations were suggested. Following Dorey [3] we will exploit a twisted chiral superfield Σ representation.

In the mirror representation [11], the superpotential is given by (see the appendix)

$$\mathcal{W}_{\text{eff}}(\Sigma) = \frac{1}{4\pi}\left(N\Sigma - \sum_{l=0}^{N-1} m_l \ln\left(\frac{2}{\mu}(\Sigma + m_l)\right)\right), \tag{7}$$

where Σ is the twisted chiral field with the lowest component σ. The vacua of the theory are the solutions of the following equation:

$$\prod_{l=0}^{N-1}(\sigma + m_l) - 1 = 0, \tag{8}$$

where we set the scale constant $\Lambda = 1$. The left-hand side of this equation is a polynomial of degree N. In the general case, it is not possible to find the roots of this equation analytically for $N \geqslant 5$. However, the Z_N symmetry of the twisted masses, given in equation (4), allows us to find the roots as

$$\sigma_k = (1 + (-m)^N)^{\frac{1}{N}}\, \mathrm{e}^{\frac{\mathrm{i}2\pi k}{N}}. \tag{9}$$

The vacua for $N = 3$ and $N = 4$ are shown in figures 3 and 4. Here we note the difference between the cases of *odd* and *even* values of N. Figure 3 is flipped with respect to figure 1, but figure 4 has the same form as figure 2. Because of this difference, we will see that CMS will be different for *odd* and *even* values of N (at finite N, not necessarily for large N).

With the explicit solution for σ_k given in equation (9), we can rewrite $\mathcal{W}_{\text{eff}}$ in the critical points (see equations (A.19) and (A.20) and [2, 3]),

$$\begin{aligned}\mathcal{W}_{\text{eff}}(\sigma_k) &= \frac{1}{4\pi}\left(N\sigma_k - \sum_{l=0}^{N-1} m\, \mathrm{e}^{\frac{\mathrm{i}2\pi l}{N}} \ln\left(\sigma_k + m\, \mathrm{e}^{\frac{\mathrm{i}2\pi l}{N}}\right)\right)\\ &= \frac{1}{4\pi}\, \mathrm{e}^{\frac{\mathrm{i}2\pi k}{N}}\left(N\sigma_0 - \sum_{l=0}^{N-1} m_l \ln(\sigma_0 + m_l)\right),\end{aligned} \tag{10}$$

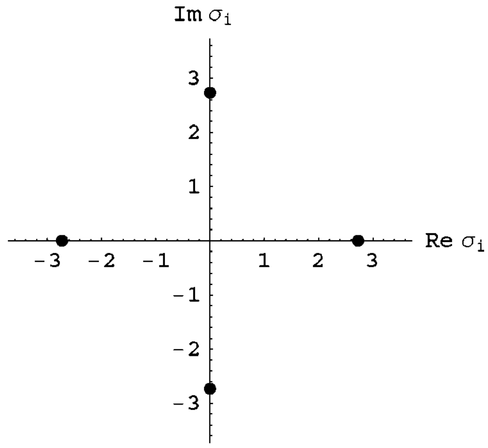

Figure 4. Z_4-symmetric vacua in the complex σ-plane.

where[3]

$$\sigma_0 = (1 + (-m)^N)^{\frac{1}{N}}. \tag{11}$$

We used the fact that $\sum_{l=0}^{N-1} m_l = 0$, and also the angular periodicity of the masses. Here we observe an important feature, namely, the index k in $\mathcal{W}_{\text{eff}}(\sigma_k)$ appears only in the phase and, as we will see, the phase factor will have no impact on the CMS consideration. Let us now consider a soliton interpolating between two vacua and carrying the topological charge $\overrightarrow{T}$. For each allowed value of the topological charge $\overrightarrow{T}$, the spectrum also includes an infinite tower of dyons with the global charge $\overrightarrow{S} = s\,\overrightarrow{T}$, where $s \in Z$. The vector $\overrightarrow{T}$ is of the form $(0, \ldots, -1, \ldots, 1, \ldots, 0)$ (for instance, for a soliton interpolating between the vacua σ_k and σ_l, $\overrightarrow{T}_k = -1$ and $\overrightarrow{T}_l = 1$). One can also introduce a topological mass vector,

$$\overrightarrow{m_D} = (\mathcal{W}_{\text{eff}}(\sigma_0), \mathcal{W}_{\text{eff}}(\sigma_1), \ldots, \mathcal{W}_{\text{eff}}(\sigma_{N-1})). \tag{12}$$

With these definitions, we can express the central charge in a compact form,

$$Z = -\mathrm{i}(\overrightarrow{m} \cdot \overrightarrow{S} + \overrightarrow{m_D} \cdot \overrightarrow{T}). \tag{13}$$

The central charge, connecting the vacua k and l, takes the form

$$\begin{aligned} Z_{kl} &= -\mathrm{i}(\overrightarrow{m} \cdot \overrightarrow{S_{kl}} + \overrightarrow{m_D} \cdot \overrightarrow{T_{kl}}) \\ &= -\mathrm{i}\,(s(m_k - m_l) + (m_{Dk} - m_{Dl})) \\ &= -\mathrm{i}m\left(\mathrm{e}^{\frac{\mathrm{i}2\pi k}{N}} - \mathrm{e}^{\frac{\mathrm{i}2\pi l}{N}}\right)\left\{s + \frac{2\mathrm{i}}{4\pi m}\left(N\sigma_0 - \sum_{j=0}^{N-1} m_j \ln(\sigma_0 + m_j)\right)\right\}. \end{aligned} \tag{14}$$

The overall factor $-\mathrm{i}m\left(\mathrm{e}^{\frac{\mathrm{i}2\pi k}{N}} - \mathrm{e}^{\frac{\mathrm{i}2\pi l}{N}}\right)$ plays no role in the determination of CMS. The condition for CMS is that the terms in the braces must have the same phase so that $|Z| = |m_{\mathcal{N}}| + |m_{\mathcal{T}}|$ is satisfied. It is clear that s is a real number. This implies that the second term must be real too on CMS. This, in turn, implies

$$\mathrm{Re}\left\{\frac{1}{2\pi m}\left(N\sigma_0 - \sum_{j=0}^{N-1} m_j\ \ln(\sigma_0 + m_j)\right)\right\} = 0. \tag{15}$$

[3] Note that σ_0 is not well defined at $(-m)^N = -1$, at which $|m| = 1$. However in the subsequent sections we will see that $|m| = 1$ is not on the CMS, so this will not affect our discussion.

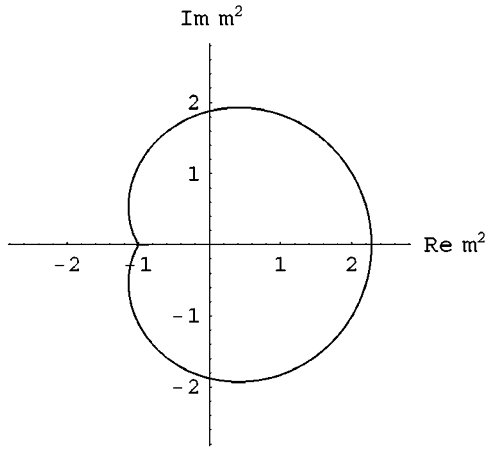

Figure 5. The contour plot for CMS for $N = 2$ in the complex m^2-plane.

Equation (15) is our basic relation determining CMS. It can be solved analytically for large N. The solution will be presented in section 3. Small-N solutions can be found numerically. For $N = 2$ this was done in [7]. For $N = 2$, equation (15) reduces to

$$\mathrm{Re}\left\{\ln\frac{1+\sqrt{1+4/m^2}}{1-\sqrt{1+4/m^2}} - 2\sqrt{1+4/m^2}\right\} = 0. \tag{16}$$

The numerical solution is reproduced in figure 5. In equation (16), we observe that the twisted mass parameter appears in the form m^2, not m. For $N = 2$ the physical sheet of the Riemann surface is the complex m^2-plane, or, equivalently half of the complex m-plane. We will see that this is a general result; for generic N the physical parameter is m^N rather than m and, therefore, it is sufficient to solve equation (15) for $|\mathrm{Arg}(m)| < \frac{\pi}{N}$, which is mapped onto the complete complex m^N-plane. To illustrate the behavior of states near CMS, let us consider an elementary state $\{T = 0, s = 1\}$ where T and s show the topological and Noether charges, respectively. From figure 5, we see that CMS cuts the real axis at about 2.3. For $\mathrm{Re}\, m^2$ slightly larger than this value, the state $\{T = 0, s = 1\}$ becomes a marginally bound state of two fundamental solitons $\{T = 1, s = 0\}$ and $\{T = -1, s = 1\}$. If we cross CMS and move to $\mathrm{Re}\, m^2$ smaller than 2.3, the interaction becomes repulsive and all the tower of excited states disappears [5, 7]. For $N = 4$ CMS is given in figure 6. (The figure is scaled by plotting $\mathrm{e}^{-4}m^4$ rather than m^4.) We see that already at $N = 4$, CMS is pretty close to a circle. It becomes perfectly circular at $N \to \infty$.

3. The large-N limit

In the remainder of the paper we will construct CMS for large N. Before delving into a detailed analysis, let us qualitatively discuss the behavior of the function in equation (15). The first term is of the order N whereas the second term, which is a sum, has oscillating terms. Although there are N terms, the result of the summation will be of the order of $|m|$ rather than of the order of N due to this oscillatory behavior. If the sum is to be of the order N, the argument of the logarithm must be exponentially small in N for at least some terms. The main strategy will be to investigate the sum to get a term of the order N, which can cancel the term $N\sigma_0$ in equation (15). In our analysis, we will constrain ourselves to the region where $|\mathrm{Arg}(m)| < \frac{\pi}{N}$

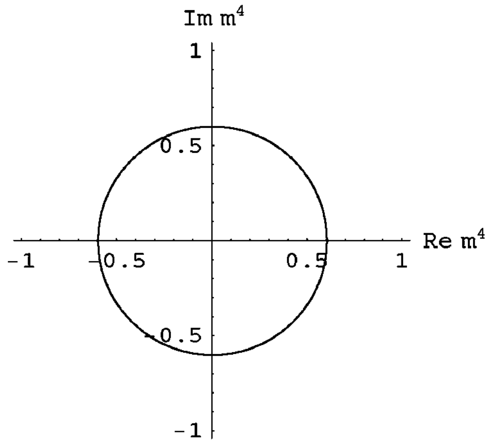

Figure 6. The contour plot for CMS for $N = 4$ in the complex m^4-plane.

in the complex m-plane. This is the region which is mapped onto the complete complex plane when we use m^N as our parameter instead of m. Due to the $(-1)^N$ term in equation (9), we see that it will be convenient to carry out the analysis for *even* and *odd* N values separately.

3.1. CMS for large and evenN

The results in the previous section show that a soliton is on the CMS if

$$\mathrm{Re}\left\{\frac{1}{m}\left(N\sigma_0 - \sum_{j=0}^{N-1} m_j \ln(\sigma_0 + m_j)\right)\right\} = 0. \tag{17}$$

To find the solution to this equation we will have to use slightly different expansions of σ_0 depending on whether $|m| > 1$, $|m| = 1$ or $|m| < 1$, which suggests separate analysis of the problem in three regions.

3.1.1. $|m| > 1$. In this case, the terms $j = \frac{N-2}{2}, \frac{N}{2}, \frac{N+2}{2}$ dominate the sum. (Actually we will find out that the central term, $j = \frac{N}{2}$, is the most dominant.) As $|m| > 1$ and $|\mathrm{Arg}(m)| < \frac{\pi}{N}$, we can use the expansion $(1 + m^N)^{\frac{1}{N}} \simeq m + \frac{1}{Nm^{N-1}}$ to get

$$\sum_{j=0}^{N-1} m_j \ln(\sigma_0 + m_j) \simeq -m \ln\left(\frac{1}{Nm^{N-2}}\right) - m\left(1 - \frac{\mathrm{i}4\pi}{N}\right)\ln\left(\frac{1}{Nm^{N-2}} - m\frac{4\pi\mathrm{i}}{N}\right)$$
$$-\left(1 + \frac{\mathrm{i}4\pi}{N}\right)\ln\left(\frac{1}{Nm^{N-2}} + m\frac{4\pi\mathrm{i}}{N}\right). \tag{18}$$

The first line is the contribution coming from the $j = \frac{N}{2}$ term, whereas the second and the third lines come from the $j = \frac{N\pm2}{2}$ terms. The first line presents the dominant term in the large-N limit. If we define the twisted mass parameter m in polar coordinates, $m = \mu\,\mathrm{e}^{\mathrm{i}\theta}$, where $\mu = |m|$, we can separate the real and imaginary parts as follows:

$$\sum_{j=0}^{N-1} m_j \ln(\sigma_0 + m_j) = -mN\left[-\frac{N-1}{N}\ln\mu + O\left(\frac{\ln N}{N}\right)\right]. \tag{19}$$

Here $O\left(\frac{\ln N}{N}\right)$ stands for terms of the order $\frac{\ln N}{N}$. Now, equation (17) implies

$$N\left(1-\frac{N-1}{N}\ln\mu\right)=0, \tag{20}$$

which in turn entails

$$\mu\to e \tag{21}$$

as $N\to\infty$ (N even). As we have discussed in the $N=2$ case, the relevant parameter is m^N rather than m. The solution described above, which is an arc of a circle of radius e and angle $|\theta|<\frac{\pi}{N}$, is mapped onto the complete circle of radius e^N in the complex m^N-plane.

3.1.2. $|m|=1$. In this case we can use the expansion $(1+m^N)^{\frac{1}{N}}\simeq 1+\frac{m}{N}\ln 2$. We can again approximate the sum by the dominant terms, but in this case the argument of the logarithm is of the order of $1/N$; hence it is impossible to cancel the leading term of the order N in equation (15) with this $\ln N$ term. Therefore, $|m|=1$ is not on the CMS for large N (N*even*).

3.1.3. $|m|<1$. In order to have a complete and careful analysis, we can subdivide this region into two parts as $|m|\to 1-\epsilon$ $(0<\epsilon\ll 1)$ and $|m|\ll 1$. The main difference between the two regions is that for the former, we can approximate the sum with the dominant terms, whereas for the latter one each term produces a contribution of the same order, and we convert the sum to an integral, which is applicable in the large-N limit. In both cases, the expansion $(1+m^N)^{\frac{1}{N}}\simeq 1+\frac{m^N}{N}$ is applicable. Let us start with the first case, $|m|\to 1-\epsilon$

$$\begin{aligned}\sum_{j=0}^{N-1} m_j\ln(\sigma_0+m_j)\simeq{}&-m\left(1-\frac{\mathrm{i}4\pi}{N}\right)\ln\left(1+\frac{m^N}{N}-m\left(1-\frac{\mathrm{i}4\pi}{N}\right)\right)\\&-m\left(1+\frac{\mathrm{i}4\pi}{N}\right)\ln\left(1+\frac{m^N}{N}-m\left(1+\frac{\mathrm{i}4\pi}{N}\right)\right)\\&-m\ln\left(1+\frac{m^N}{N}-m\right).\end{aligned} \tag{22}$$

Again, inspecting the arguments of the logarithms, we note that we end up with terms of the order $\ln N$, which cannot cancel the leading term in equation (15). Therefore, we conclude that $|m|\to 1-\epsilon$ does not belong to CMS. In the second case, $|m|\ll 1$, we need to change our strategy. The sum cannot be approximated by a few dominant terms, as they contribute almost equally. So we convert the sum to a corresponding integral,

$$\begin{aligned}\sum_{j=0}^{N-1} m_j\ln(\sigma_0+m_j)&\simeq\frac{N}{2\pi\mathrm{i}}\int_0^{\frac{2\pi(N-1)}{N}}\mathrm{i}m\ln(\sigma_0+m\,\mathrm{e}^{\mathrm{i}x})\,\mathrm{d}x\\&\simeq -m\ln(1+m).\end{aligned} \tag{23}$$

This sum is of the order of $m\ln(m+1)\simeq O(1)$, which cannot cancel the leading term in equation (15). Therefore, we conclude that $|m|\ll 1$ does not belong to CMS either. Combining all the results above, we see that in the large-N limit (with N *even*),

$$\mu^N(\theta)=\mathrm{e}^N,\quad -\pi<\theta<\pi, \tag{24}$$

which means that CMS are circles of the radius e^N in the complex m^N-plane (in the complex m-plane we have CMS at $|m|=e\Lambda$).

3.2. CMS for large and odd N

The analysis for *odd N* is slightly different than for the *even N* case. For *odd N*, we have $\sigma_0 = (1-m^N)^{\frac{1}{N}}$. The large-$N$ expansion will have an extra phase factor compared to the *even N* case. We will present the analysis for $|m| > 1$, which will produce the CMS for *odd N*.

3.2.1. $|m| > 1$. The expansion of $(1-m^N)^{\frac{1}{N}}$ depends on the phase of m. For $-\frac{\pi}{N} < \mathrm{Arg}(m) < 0$, we can use the expansion $(1-m^N)^{\frac{1}{N}} \simeq \mathrm{e}^{\mathrm{i}\frac{\pi}{N}}\left(m - \frac{1}{m^{N-1}}\frac{1}{N}\right)$. In this case, the $j = \frac{N+1}{2}$ term dominates the sum. With this expansion, equation (17) reads

$$\mathrm{Re}\left\{N\,\mathrm{e}^{\mathrm{i}\frac{\pi}{N}} + \mathrm{e}^{\mathrm{i}\frac{\pi}{N}}\ln\left(-m^{-N+1}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{N}}\right)\right\} = 0, \tag{25}$$

where we kept only $O(N)$ terms. In the polar coordinates, $m = \mu\,\mathrm{e}^{\mathrm{i}\theta}$ with $-\frac{\pi}{N} < \theta < 0$, equation (25) reduces to

$$N - (1-N)\ln\mu + \frac{\pi}{N}(\pi + N\theta) = 0. \tag{26}$$

This equation has the following solution:

$$\mu(\theta) = \mathrm{e}^{1+\frac{\pi}{N^2}(\pi+N\theta)}, \qquad -\frac{\pi}{N} < \theta < 0. \tag{27}$$

At $N \to \infty$, this result reduces to the *even-N* result, which is expected, of course. For $0 < \mathrm{Arg}(m) < \frac{\pi}{N}$, we can use the expansion $(1-m^N)^{\frac{1}{N}} \simeq \mathrm{e}^{-\mathrm{i}\frac{\pi}{N}}\left(m - \frac{1}{m^{N-1}}\frac{1}{N}\right)$. Now the $j = \frac{N-1}{2}$ term dominates the sum. Making the same steps we get

$$\mu(\theta) = \mathrm{e}^{1+\frac{\pi}{N^2}(\pi-N\theta)}, \qquad 0 < \theta < \frac{\pi}{N}. \tag{28}$$

We can combine both results as follows,

$$\mu(\theta) = \mathrm{e}^{1+\frac{\pi}{N^2}(\pi-N|\theta|)}, \qquad -\frac{\pi}{N} < \theta < \frac{\pi}{N}. \tag{29}$$

As in the *even-N* case, we need to map this solution onto the complex m^N-plane. Then CMS becomes

$$\mu^N(\theta) = \mathrm{e}^{N+\frac{\pi}{N}(\pi-|\theta|)}, \qquad -\pi < \theta < \pi. \tag{30}$$

This result reduces to the *even-N* result at $N \to \infty$. Collecting all the results for *even* and *odd N*, we conclude that CMS are circles of radius e^N in the complex m^N-plane. Numerical solutions for CMS for $N = 10$ and $N = 11$ are given in figures 7 and 8. We plotted $\mathrm{e}^{-N}m^N$ rather than m^N so that the radius becomes unity in the large-N limit. We note that CMS for $N = 10$ is a circle and its radius is slightly less than unity, which is the case expected at $N \to \infty$. The deviation from circle is more pronounced for *odd N*, as seen in figure 8. This behavior is consistent with the large-N limit given in equation (30), from which we see that the radial coordinate depends on the angle. At $\theta = 0$, the radius is enlarged by a factor of $\mathrm{e}^{\frac{\pi^2}{N}}$, which is about 2.45 at $N = 11$. The *even* $-N$ result, equation (20), gets an enlargement factor of $\mathrm{e}^{\frac{1}{N}}$ for any angle, which is close to unity at $N = 10$. This explains why *even* $-N$ results converge to $N \to \infty$ limit faster than *odd* $-N$ results.

4. Summary

In this paper, we discussed CMS in the $\mathcal{N} = (2,2)$ supersymmetric $CP(N-1)$ model with the Z_N-symmetric twisted masses. The CMS condition is given by equation (15). The solution to this equation is given in the complex m^N-plane. We show that, for large values of N, CMS are

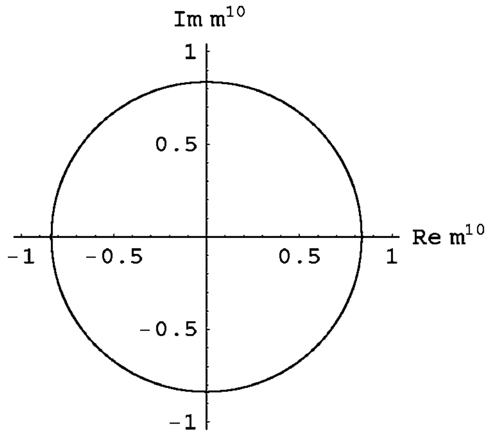

Figure 7. The contour plot for CMS for $N = 10$ in the complex m^{10}-plane.

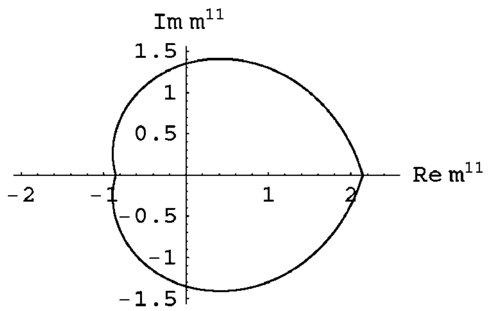

Figure 8. The contour plot for CMS for $N = 11$ in the complex m^{11}-plane.

circles of the radius of e^N in the complex m^N-plane, which corresponds to $|m| = e$. This result is approximate up to terms of the order $\frac{\ln N}{N}$ which die off rather slowly. This is the reason to have different CMS curves for *odd* and *even* N, for small N, as formulated in equations (24) and (30) respectively. If we consider the $N \to \infty$-limit, at which $\frac{\ln N}{N} \ll 1$, CMS are circles of the radius e^N for *odd* and *even* N. It is curious to note that in a non-supersymmetric $CP(N-1)$ model, the curve of the phase transition is also circular at the large N [6].

Acknowledgments

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Appendix

A.1. $N = 2$ Supersymmetry in two dimensions

$N = 1$ supersymmetry algebra [12] has an $U(1)R$-symmetry under which the left-handed supercharges have charge -1 and the right-handed ones have charge $+1$. One can obtain $N = 2$

supersymmetry in two dimensions by dimensional reduction from $N = 1$ supersymmetry in four dimensions [8]. Eliminating the dependence of fields on two coordinates, say x^2 and x^3, we get the two-dimensional Lorentz group and an internal symmetry group associated with the rotations around the eliminated coordinates. This internal symmetry is called the $U(1)_A$ symmetry. With this reduction, a left-handed spinor in four dimensions becomes the Dirac spinor in two dimensions, which consists of one left- and one right-handed spinor with the opposite $U(1)_A$ charges. The supercharges of the four-dimensional theory reduce to two Dirac spinors $Q_{L,R}$ and $\bar{Q}_{L,R}$. L, R shows the two-dimensional chirality whereas *bar* shows the four-dimensional chirality. The Dirac spinors $Q_{L,R}$ and $\bar{Q}_{L,R}$ carry the $U(1)_A$ charges $-1, +1$ and $+1, -1$, respectively. They are Hermitian conjugate to each other, $(Q_{L,R})^\dagger = \bar{Q}_{L,R}$. The anticommutation relations in two dimensions can be written as

$$\begin{aligned} &\{Q_L, \bar{Q}_L\} = 2(H + P), \\ &\{Q_R, \bar{Q}_R\} = 2(H - P), \\ &Q_L^2 = Q_R^2 = \bar{Q}_L^2 = \bar{Q}_R^2 = 0, \end{aligned} \tag{A.1}$$

where H and P are the Hamiltonian and the momentum operators. All the other commutators vanish unless there are central charges, of which we will speak later. In two dimensions, the $U(1)R$-symmetry of four-dimensional theory appears as another internal symmetry which is called the $U(1)_V$-symmetry. Under the $U(1)_V$ symmetry, the supercharges $Q_{L,R}$ and $\bar{Q}_{L,R}$ have the charges $-1, -1$ and $+1, +1$, respectively. So, in two dimensions, there are two $U(1)$ R-symmetry groups, $U(1)_V$ and $U(1)_A$. The supercharges can be grouped as

$$\begin{matrix} Q_R & \bar{Q}_L \\ Q_L & \bar{Q}_R, \end{matrix} \tag{A.2}$$

where the first (second) line has the $U(1)_A$ charge $+1$ (-1) and the left (right) column has the $U(1)_V$ charge -1 $(+1)$. An important property of $N = 2$ supersymmetry in two dimensions is that it is possible to have a field Σ which obeys

$$\bar{\mathcal{D}}_L \Sigma = 0 = \mathcal{D}_R \Sigma, \tag{A.3}$$

(compare with the chiral field which obeys $\bar{\mathcal{D}}_L \Phi = 0 = \bar{\mathcal{D}}_R \Phi$) which is called the *twisted chiral field*. Using the Bianchi identities, it is easy to get Σ,

$$\Sigma = \tfrac{1}{2}\{\bar{\mathcal{D}}_L, \mathcal{D}_R\}. \tag{A.4}$$

A.2. Central extension and mirror symmetry

The $N = 2$ supersymmetry algebra can be extended by the inclusion of central charges which are associated with the topological charge of the soliton sectors [14]. As the central term should commute also with R-symmetry, the central extension breaks the $U(1)_V$ or/and the $U(1)_A$ symmetries.

For instance, consider a massive theory in which the $U(1)_V$ symmetry is broken by a superpotential. Due to this central extension, we have nonzero (anti)commutation relations in addition to equation (A.1),

$$\begin{aligned} &\{Q_L, Q_R\} = 2Z, \qquad \{\bar{Q}_L, \bar{Q}_R\} = 2Z^*, \\ &\{Q_L, \bar{Q}_R\} = 0, \qquad \{\bar{Q}_L, Q_R\} = 0, \\ &[F_A, Q_L] = -Q_L, \quad [F_A, Q_R] = Q_R, \quad [F_A, \bar{Q}_L] = \bar{Q}_L, \quad [F_A, \bar{Q}_R] = -\bar{Q}_R, \end{aligned} \tag{A.5}$$

where F_{A} denotes the generator of the $U(1)_{\mathrm{A}}$ R-symmetry. Using the (anti)commutation relations in equations (A.1) and (A.5) we observe that the mass of the particle in a sector with central charge Z is bounded from below by

$$M \geqslant |Z|. \tag{A.6}$$

One can derive this result by calculating the anticommutator of the operators $(H-P)Q_L - Z\bar{Q}_R$ and its Hermitian conjugate, which is positive semi-definite by construction. The equality in equation (A.6) is satisfied if

$$(H-P)Q_L = Z\bar{Q}_R, \tag{A.7}$$

which is called the *BPS condition*. Consider eigenstates of energy and momentum. For these eigenstates, equation (A.7) and its Hermitian conjugate imply that Q_L and $\bar{Q}_L$ are proportional to $\bar{Q}_R$ and Q_R, respectively. So the supersymmetry multiplet is shortened. This is called a *BPS multiplet*.

We can also consider a theory in which the $U(1)_{\mathrm{A}}$ symmetry is broken. In this case, in addition to equation (A.1), the algebra reads

$$\begin{aligned}
&\{Q_L, Q_R\} = 0, \qquad \{\bar{Q}_L, \bar{Q}_R\} = 0,\\
&\{Q_L, \bar{Q}_R\} = 2\widetilde{Z}, \qquad \{\bar{Q}_L, Q_R\} = 2\widetilde{Z}^*,\\
&[F_{\mathrm{V}}, Q_L] = -Q_L, \quad [F_{\mathrm{V}}, Q_R] = -Q_R, \quad [F_{\mathrm{V}}, \bar{Q}_L] = \bar{Q}_L, \quad [F_{\mathrm{V}}, \bar{Q}_R] = \bar{Q}_R,
\end{aligned} \tag{A.8}$$

where F_{V} denotes the generator of the $U(1)_{\mathrm{V}}$ R-symmetry. It is interesting to note that the (anti)commutation relations would be the same in the theories with broken $U(1)_{\mathrm{V}}$ and $U(1)_{\mathrm{A}}$ symmetry if

$$F_{\mathrm{A}} \longleftrightarrow F_{\mathrm{V}} \tag{A.9}$$

$$Q_R \longleftrightarrow \bar{Q}_R. \tag{A.10}$$

This automorphism of the $N = 2$ supersymmetry algebra is called the *mirror symmetry* [11].

A.3. CP(N-1) models with twisted masses

Consider a superrenormalizable $U(1)$ theory with N chiral superfields Φ_i with +1 charge, a gauge superfield and the corresponding field strength Σ, which is a twisted chiral superfield. The kinetic term and the interaction term are written as a D-term in $\mathcal{N} = 2$ superspace,

$$\mathcal{L}_D = \int \mathrm{d}^4\theta \left(\sum_{i=1}^{N} \bar{\Phi}_i \, \mathrm{e}^{2V} \Phi_i - \frac{1}{2\,\mathrm{e}^2} \mathrm{Tr}\, \bar{\Sigma}\Sigma \right). \tag{A.11}$$

It is convenient to combine the Fayet–Iliopoulos term and the topological θ term in the twisted F term with the Lagrangian,

$$\mathcal{L}_F = \int \mathrm{d}^2\theta \, \mathcal{W}(\Sigma) + \mathrm{H.C.} \tag{A.12}$$

The twisted superpotential is

$$\mathcal{W}(\Sigma) = \frac{\mathrm{i}\tau\Sigma}{2}, \tag{A.13}$$

where $\tau = \mathrm{i}r + \frac{\theta}{2\pi}$ [8]. Now let us consider renormalizability of the theory without the twisted masses. Gauge theories in two dimensions are superrenormalizable. In our case, the only divergence comes from a one-loop diagram and it can be absorbed into redefinition of the FI parameter as follows,

$$r(\mu) = r_0 - \frac{N}{4\pi}\ln\left(\frac{M_{\mathrm{UV}}^2}{\mu^2}\right), \tag{A.14}$$

where M_{UV} is the ultra violet cut-off and μ is the RG subtraction scale. With the renormalized FI term, the superpotential in equation (A.13) reads

$$\mathcal{W}_{\mathrm{eff}}(\Sigma) = \frac{\mathrm{i}}{2}\Sigma\left(\hat{\tau} - \frac{N}{2\pi\mathrm{i}}\ln\left(\frac{2\Sigma}{\mu}\right)\right), \tag{A.15}$$

where $\hat{\tau} = \mathrm{i}r(\mu) + \frac{\theta}{2\pi} + n^*$ with n^* chosen to minimize the potential energy. The condition for a supersymmetric vacuum is

$$\frac{\partial}{\partial\sigma}\mathcal{W}_{\mathrm{eff}} = 0, \tag{A.16}$$

where σ is the lowest component of Σ. Equation (A.16) has the following solution,

$$\sigma^N = \left(\frac{\mu}{2}\right)^N \mathrm{e}^{2\pi\mathrm{i}\tau(\mu)} \equiv \tilde{\Lambda}^N, \qquad \sigma_k = \tilde{\Lambda}\,\mathrm{e}^{\frac{2\pi\mathrm{i}k}{N}} \tag{A.17}$$

The mass of the soliton interpolating between vacua k and l is given by

$$M = 2|\mathcal{W}(\sigma_k) - \mathcal{W}(\sigma_l)|. \tag{A.18}$$

When we include the twisted masses in our theory, we have to modify equation (A.15). Now the superpotential reads

$$\mathcal{W}_{\mathrm{eff}}(\Sigma) = \frac{\mathrm{i}}{2}\left(\Sigma\hat{\tau} - \frac{1}{2\pi\mathrm{i}}\sum_{l=0}^{N-1}(\Sigma + m_l)\ln\left(\frac{2}{\mu}(\Sigma + m_l)\right)\right). \tag{A.19}$$

$\hat{\tau}$ in this equation is determined by setting $\frac{\partial}{\partial\sigma}W_{\mathrm{eff}} = 0$. Imposing this condition for $\hat{\tau}$, we get,

$$\mathcal{W}_{\mathrm{eff}}(\Sigma) = \frac{1}{4\pi}\left(N\Sigma - \sum_{l=0}^{N-1} m_l\ln\left(\frac{2}{\mu}(\Sigma + m_l)\right)\right). \tag{A.20}$$

This is the main formula that we will use to extract the topological masses of the solitons (for a pedagogical introduction to solitons see [13]). For this aim, we will also need the supersymmetric vacua as the solitons are the objects interpolating between different supersymmetric vacua. The equation for supersymmetric vacua is given by

$$\prod_{l=0}^{N-1}(\sigma + m_l) - \tilde{\Lambda}^N = 0, \tag{A.21}$$

which gives equation (A.17) if the twisted masses are all vanishing. Calling the roots of this polynomial equation σ_l, we see that there are N supersymmetric vacua with $\sigma = \sigma_l$. The BPS spectrum includes solitons interpolating between different vacua, and carrying topological charges T_i as well as elementary particles carrying global $U(1)$ charges S_i. For each pair of supersymmetric vacua, there exists a soliton interpolating between them, which means that there are $\frac{N(N-1)}{2}$ solitons carrying topological charge $\overrightarrow{T}$. For each allowed value of the topological charge $\overrightarrow{T}$, the spectrum also includes an infinite tower of dyons with the global charge $\overrightarrow{S} = s\overrightarrow{T}$, where $s \in Z$. One can also introduce topological mass vector,

$$\overrightarrow{m_D} = (\mathcal{W}_{\mathrm{eff}}(\sigma_0), \mathcal{W}_{\mathrm{eff}}(\sigma_1), \ldots, \mathcal{W}_{\mathrm{eff}}(\sigma_{N-1})). \tag{A.22}$$

With these definitions, we can express the central charge as

$$Z = -i(\overrightarrow{m} \cdot \overrightarrow{S} + \overrightarrow{m_D} \cdot \overrightarrow{T}). \tag{A.23}$$

In order to have a BPS state to decay into its constituents, its mass must be equal to the sum of the masses of its constituents,

$$|Z| = |\overrightarrow{m} \cdot \overrightarrow{S}| + |\overrightarrow{m_D} \cdot \overrightarrow{T}|, \tag{A.24}$$

which is equivalent to requiring that each term in Z to have the same phase. This is the condition that determines CMS.

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